Unit D3

Series

Introduction

In this unit you will study *infinite series*. Informally, an infinite series is the sum of infinitely many numbers such as

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots.$$

Although in everyday language the words 'sequence' and 'series' are often used interchangeably, in mathematics they represent distinct, but related, concepts.

As with sequences, an infinite series may converge or diverge. The unit begins with a formal definition of a convergent infinite series, illustrated with several examples. You will then learn how to use various tests to determine whether or not a series converges.

Because the study of series involves studying related sequences, this unit depends heavily on the ideas and results of Unit D2 *Sequences*; therefore, before studying this unit you should make sure that you understand Unit D2, Sections 1, 2 and 3, and that you are familiar with Sections 4 and 5.

When working with series, we often need to determine whether a related sequence converges and, if it does, find the value of its limit. In this unit we do not always give as much detail in such calculations as we did in Unit D2. As is the case throughout this module, the amount of detail given in solutions to the exercises and worked exercises indicates the amount you should give in your own solutions.

1 Introducing series

In this section you will see how a convergent infinite series can be defined formally in terms of convergent sequences. You will meet several examples of such series and discover various properties that all convergent series have in common.

1.1 What is a convergent series?

We begin by considering a paradox of Zeno.

> The Ancient Greek philosopher and mathematician Zeno lived and worked in Elea in southern Italy during the 5th century BCE. Zeno proposed a number of paradoxes of the infinite, which have intrigued succeeding generations. His paradoxes include 'The Flying Arrow' and 'Achilles and the Tortoise'.

In one of his paradoxes, Zeno claimed that it is impossible for an object to travel a given distance, since it must first travel half the distance, then half of the remaining distance, then half of what remains, and so on. There must always remain some distance left to travel, so the journey cannot be completed.

This paradox relies partly on the intuitive feeling that it is impossible to add up an infinite number of positive quantities and obtain a finite answer. However, the illustration of the paradox in Figure 1 suggests that, in this case, a finite answer is certainly plausible.

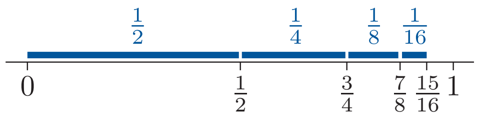


Figure 1 The distances in Zeno's paradox

The distance from 0 to 1 can be split up into the infinite sequence of distances $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, each distance being half of the preceding one, so it seems reasonable to write

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

We now give a justification for this statement. Let s_n be the sum of the first n terms on the left-hand side. We call this the nth *partial sum*. Then

$$s_1 = \frac{1}{2},$$
$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$
$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}.$$

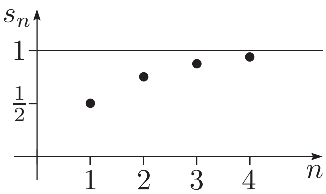


Figure 2 The sequence (s_n) of partial sums

The sequence diagram for the sequence (s_n) of partial sums is shown in Figure 2.

To obtain the nth partial sum, we use the formula for the sum of a finite geometric series. Here is a reminder of the formula.

Sum of a finite geometric series

The geometric series with first term a, common ratio $r \neq 1$ and n terms has the sum

$$a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

By applying this formula in the case that $a = r = \frac{1}{2}$, we obtain

$$\begin{aligned} s_n &= \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^n \\ &= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} \\ &= \frac{\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right)}{1 - \frac{1}{2}} = 1 - \left(\tfrac{1}{2}\right)^n. \end{aligned}$$

The sequence $\left(\left(\frac{1}{2}\right)^n\right)$ is a basic null sequence, so

$$\lim_{n\to\infty} s_n = 1 - \lim_{n\to\infty} \left(\tfrac{1}{2}\right)^n = 1.$$

It is this precise mathematical statement that $s_n \to 1$ as $n \to \infty$ which justifies our informal statement earlier that

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

We now use this approach to define a *convergent infinite series*.

Definitions

Let (a_n) be a sequence. Then the expression

$$a_1 + a_2 + a_3 + \cdots$$

is an **infinite series**, or simply a **series**.

We call a_n the ***n*th term** of the series and

$$s_n = a_1 + a_2 + \cdots + a_n$$

the ***n*th partial sum** of the series.

The series is **convergent** with **sum** s (or **converges to** s) if its sequence (s_n) of partial sums converges to s. In this case, we write

$$a_1 + a_2 + a_3 + \cdots = s.$$

The series **diverges**, or is **divergent**, if the sequence (s_n) diverges.

Notice that the sum of a convergent infinite series is the limit of the sequence of its partial sums. Thus we can prove results about a series by applying known results about sequences to its partial sums (s_n).

Remarks

1. When you have a series $a_1 + a_2 + \cdots$, it is important to distinguish between the sequence of terms

   $$(a_n) = a_1,\ a_2,\ a_3, \ldots$$

   and the sequence of partial sums

   $$(s_n) = a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3, \ldots .$$

   It may help you remember the difference if you think of s_n as the *sum* of the series up to and including the nth term (s standing for 'sum'), and a_n as the amount that is *added* to the series by the nth term (a standing for 'added').

   It may also be helpful to think of the series as a 'snake' which

   - has length s_n on its nth birthday, and
   - grows by the length a_n in its nth year.

This is illustrated in Figure 3.

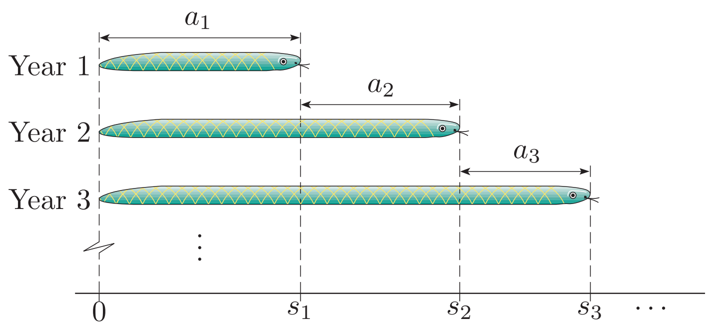


Figure 3 The sequences (s_n) and (a_n)

This picture has its limitations, however; for example, it assumes that the snake lives forever. Also the terms a_n need not be positive, so the snake may shrink or even have negative length!

2. Sometimes the first term in a series may not be a_1; for example, it might instead be a_0 or a_3. However, when a series begins with a term other than a_1, we still calculate the nth partial sum s_n by adding all the terms up to and including a_n.

   For example, if the first term of a series is a_0, then the nth partial sum is

   $$s_n = a_0 + a_1 + \cdots + a_n.$$

   Hence, in the case of a series starting with a_0, there is a 0th partial sum $s_0 = a_0$, and the nth partial sum is the sum of $n + 1$ terms.

3. Note that changing, deleting or adding a *finite* number of terms does not affect the convergence of a series, but may affect its sum. For example, the series

   $$1 + 2 + 3 + 4 + 5 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

   is convergent with sum $1 + 2 + 3 + 4 + 5 + 1 = 16$.

Worked Exercise D28

For each of the following series, calculate the nth partial sum and determine whether the series is convergent or divergent.

(a) $1 + 1 + 1 + \cdots$ (b) $\frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$ (c) $2 + 4 + 8 + \cdots$

Solution

(a) In this case,

$$s_n = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n.$$

So $s_n \to \infty$ as $n \to \infty$, and this series is divergent.

(b) Here we can use the formula for the sum of a finite geometric series to obtain the nth partial sum.

Putting $a = r = \frac{1}{3}$ in the formula for the sum of a finite geometric series, we obtain

$$s_n = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{3}\right)^n$$
$$= \frac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}} = \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^n\right).$$

Since $\left(\left(\frac{1}{3}\right)^n\right)$ is a basic null sequence,

$$\lim_{n\to\infty} s_n = \tfrac{1}{2},$$

so this series is convergent with sum $\frac{1}{2}$.

(c) In this case, the formula for the sum of a finite geometric series with $a = r = 2$ gives

$$s_n = 2 + 4 + 8 + \cdots + 2^n = \frac{2(1-2^n)}{1-2} = 2^{n+1} - 2.$$

So $s_n \to \infty$ as $n \to \infty$, and this series is divergent.

The next worked exercise shows that infinite series need to be manipulated with care.

Worked Exercise D29

Explain why the following proof is incorrect.

Claim (incorrect!)

$$2 + 4 + 8 + \cdots = -2.$$

Proof (incorrect!) Let

$$2 + 4 + 8 + \cdots = s.$$

Multiplying through by $\frac{1}{2}$ gives

$$1 + 2 + 4 + 8 + \cdots = \tfrac{1}{2}s,$$

which can be written as

$$1 + s = \tfrac{1}{2}s.$$

So $\frac{1}{2}s = -1$ and hence

$$s = -2.$$ ■

Solution

The problem with this 'proof' is that it is based on the assumption that s is a finite number. In fact, as we saw in Worked Exercise D28(c), the series

$$2 + 4 + 8 + \cdots$$

is divergent and its partial sums tend to infinity. Thus we cannot perform arithmetic operations with s and so the proof is not valid.

We can avoid reaching absurd conclusions such as that in Worked Exercise D29 by performing arithmetic operations only with infinite series which we know to be *convergent*. It is therefore very important to be able to check whether or not a given series is convergent. You will meet many ways of doing this for different types of series in this unit.

Sigma notation

First, however, we show how to use *sigma notation* as a convenient way to represent infinite series.

From your previous studies you will be familiar with the use of sigma notation as a shorthand way of writing *finite* sums. For example, instead of $a_1 + a_2 + \cdots + a_{10}$, we can write

$$\sum_{n=1}^{10} a_n,$$

where the symbol $\sum$ is the Greek upper-case letter sigma (standing for 'sum') and the subscript n takes all integer values from 1 to 10 inclusive.

This notation can readily be adapted to represent infinite series. Instead of $a_1 + a_2 + a_3 + \cdots$, we write

$$\sum_{n=1}^{\infty} a_n,$$

which is read as 'sigma, $n = 1$ to infinity, a_n', or 'the sum from $n = 1$ to infinity of a_n'. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

Remarks

1. An alternative layout for sigma notation is $\sum_{n=1}^{\infty} a_n$. We sometimes use the simpler notation $\sum a_n$ to denote a general infinite series with terms a_n.
2. Note that there is no term a_∞ in the series $\sum_{n=1}^{\infty} a_n$. The symbol ∞ is here used to mean that the subscript n takes every integer value greater than or equal to 1.
3. When using sigma notation to represent the nth partial sum s_n of a series, we use another letter for the subscript of the terms to avoid n having two different meanings in the same expression; we may write

   $$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

   For example, we write

   $$s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}.$$

   The letters i, j, k, l, m, n, p and q are commonly used for subscript variables.
4. If a series begins with a term other than a_1, then we adapt the notation appropriately; for instance,

   $$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots \quad \text{or} \quad \sum_{n=3}^{\infty} a_n = a_3 + a_4 + a_5 + \cdots .$$

   For example, we write

   $$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots .$$

Exercise D41

For each of the following series, calculate the nth partial sum s_n and determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(-\tfrac{1}{3}\right)^n$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ (c) $\displaystyle\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n$

The series considered so far in this section are all geometric series. In general, the **(infinite) geometric series** with first term a and common ratio r is

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots .$$

Note that it is conventional to regard the term ar^n as the nth term of a geometric series, which means that the summation goes from $n = 0$ to ∞. (We did not adopt this convention for some of the geometric series you have met so far, but we will do so from now on.)

The following theorem enables us to decide whether any given geometric series is convergent or divergent.

Theorem D24 Geometric series

(a) If $|r| < 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ is convergent, with sum $\dfrac{a}{1-r}$.

(b) If $|r| \geq 1$ and $a \neq 0$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ is divergent.

Proof (a) If $r \neq 1$, then the nth partial sum s_n is given by the formula for the sum of a finite geometric series with $n+1$ terms, so

$$s_n = a + ar + ar^2 + \cdots + ar^n = \frac{a(1-r^{n+1})}{1-r}. \tag{1}$$

Now, if $|r| < 1$, then (r^n) is a basic null sequence, so

$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{a(1-r^{n+1})}{1-r}\\ &= \frac{a}{1-r}\left(1 - \lim_{n\to\infty} r^{n+1}\right) = \frac{a}{1-r},\end{aligned}$$

by the Combination Rules for sequences that you met in Unit D2. Thus, if $|r| < 1$, then

$$\sum_{n=0}^{\infty} ar^n \text{ is convergent, with sum } \frac{a}{1-r}.$$

(b) We deal separately with the cases $r = \pm 1$ and $|r| > 1$.

We saw in Worked Exercise D28(a) and Exercise D41(b) that a geometric series with $r = \pm 1$ and $a = 1$ is divergent. The same arguments can be used to show that any geometric series with $r = \pm 1$ is divergent, whatever the value of a.

If $|r| > 1$, then (s_n) is given by equation (1) and $|r|^{n+1} \to \infty$ as $n \to \infty$, so the sequence (s_n) is unbounded and hence divergent.

Thus, if $|r| \geq 1$, then $\displaystyle\sum_{n=0}^{\infty} ar^n$ is divergent. ■

1.2 Telescoping series

Geometric series are easy to deal with because there is a formula for the nth partial sum s_n. The next exercise concerns another series for which we can calculate a formula for s_n.

Exercise D42

Calculate the first four partial sums of the following series, giving your answers as fractions:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots .$$

The partial sums obtained in Exercise D42 suggest the general formula

$$s_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

This formula can be proved by using the identity

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \quad \text{for } n = 1, 2, \ldots,$$

which implies that

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right).$$

Thus

$$\begin{aligned} s_n &= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)n} + \frac{1}{n(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1}. \end{aligned}$$

(This series is said to be *telescoping* because of the cancellation of the adjacent terms $-\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, \frac{1}{3}, \ldots$.)

Since

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1,$$

we deduce that the given series is convergent, with sum

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Exercise D43

Find the nth partial sum s_n of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$, using the identity

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}, \quad \text{for } n = 1, 2, \ldots.$$

Deduce that this series is convergent and find its sum.

1.3 Combination Rules for series

You have already seen that performing arithmetic operations on the divergent series $2 + 4 + 8 + \cdots$ can lead to absurd conclusions. However, we can perform arithmetic operations on convergent series. The following result shows that there are Combination Rules for convergent series, which follow directly from the Combination Rules for sequences.

Theorem D25 Combination Rules for convergent series

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n = s$ and $\displaystyle\sum_{n=1}^{\infty} b_n = t$. Then

Sum Rule $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = s + t$

Multiple Rule $\displaystyle\sum_{n=1}^{\infty} \lambda a_n = \lambda s$, for any real number λ.

Proof Consider the sequences of partial sums (s_n) and (t_n), where

$$s_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad t_n = \sum_{k=1}^{n} b_k.$$

We know that $s_n \to s$ as $n \to \infty$ and $t_n \to t$ as $n \to \infty$.

Sum Rule The nth partial sum of the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is

$$\begin{aligned}\sum_{k=1}^{n}(a_k + b_k) &= (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n)\\ &= (a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)\\ &= s_n + t_n.\end{aligned}$$

By the Sum Rule for sequences,

$$\lim_{n\to\infty}(s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t,$$

so the sequence $(s_n + t_n)$ of partial sums of $\sum_{n=1}^{\infty}(a_n + b_n)$ has limit $s + t$. Hence this series is convergent and

$$\sum_{n=1}^{\infty}(a_n + b_n) = s + t.$$

Multiple Rule The nth partial sum of the series $\sum_{n=1}^{\infty}\lambda a_n$ is

$$\begin{aligned}\sum_{k=1}^{n}\lambda a_k &= \lambda a_1 + \lambda a_2 + \cdots + \lambda a_n\\ &= \lambda(a_1 + a_2 + \cdots + a_n)\\ &= \lambda s_n.\end{aligned}$$

By the Multiple Rule for sequences,

$$\lim_{n\to\infty}(\lambda s_n) = \lambda \lim_{n\to\infty} s_n = \lambda s,$$

so the sequence (λs_n) of partial sums of $\sum_{n=1}^{\infty}\lambda a_n$ has limit λs. Hence this series is convergent and

$$\sum_{n=1}^{\infty}\lambda a_n = \lambda s.$$

■

Worked Exercise D30

Prove that the following series is convergent and calculate its sum:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n} + \frac{3}{n(n+1)}\right).$$

Solution

This series is of the form $\sum_{n=1}^{\infty}(a_n + 3b_n)$, where

$$a_n = \frac{1}{2^n} \quad\text{and}\quad b_n = \frac{1}{n(n+1)}.$$

We studied the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ earlier in the section.

At the beginning of Subsection 1.1 we saw that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ is convergent, with sum 1.}$$

Also, in Subsection 1.2, we saw that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ is convergent, with sum 1.}$$

To deduce the sum of the given series from these results, we use both the Sum Rule and the Multiple Rule.

Hence, by the Combination Rules,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{3}{n(n+1)}\right) \text{ is convergent, with sum } 1 + (3 \times 1) = 4.$$

Exercise D44

Prove that the following series is convergent and calculate its sum:

$$\sum_{n=1}^{\infty} \left(\left(-\tfrac{3}{4}\right)^n - \frac{2}{n(n+1)}\right).$$

We conclude this subsection with the following corollary to the Multiple Rule for convergent series, which tells us that a non-zero multiple of a *divergent* series is also divergent. (Remember, though, that you can *never* perform arithmetic operations with divergent series.)

Corollary D26 Multiple Rule for divergent series

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent and that λ is a non-zero real number.

Then $\displaystyle\sum_{n=1}^{\infty} \lambda a_n$ is divergent.

Proof We use proof by contradiction. Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent, and that λ is some non-zero real number such that $\displaystyle\sum_{n=1}^{\infty} \lambda a_n$ is convergent.

Then it follows from the Multiple Rule for convergent series with multiple $1/\lambda$ that

$$\sum_{n=1}^{\infty} a_n = \frac{1}{\lambda}\sum_{n=1}^{\infty} \lambda a_n \text{ is convergent.}$$

However, this is a contradiction since we know that $\sum_{n=1}^{\infty} a_n$ is divergent. So our original assumption was wrong and we must in fact have

$$\sum_{n=1}^{\infty} \lambda a_n \text{ is divergent, for any non-zero real number } \lambda.$$

■

1.4 Non-null Test

For all the infinite series we have so far considered, it is possible to derive a simple formula for the nth partial sum. For many series, however, this is difficult or even impossible.

Nevertheless, it may still be possible to decide whether such series are convergent or divergent by applying various tests. Our first test arises from the following result.

Theorem D27

If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then its sequence of terms (a_n) is a null sequence.

Proof Let

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

denote the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$. Because $\sum_{n=1}^{\infty} a_n$ is convergent, we know that (s_n) is a convergent sequence, with limit s, say.

We want to deduce that (a_n) is null. To do this, we note that

$$a_n = s_n - s_{n-1}, \quad \text{for } n \geq 2.$$

Thus, by the Combination Rules for convergent sequences,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1}.$$

The sequence $(s_{n-1})_2^{\infty}$ is the same as the sequence $(s_n)_1^{\infty}$, so $s_{n-1} \to s$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} a_n = s - s = 0$$

so (a_n) is a null sequence, as required. ■

The following test for divergence is an immediate corollary of Theorem D27.

Corollary D28 Non-null Test

If (a_n) is not a null sequence, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Although it is sometimes obvious that a sequence (a_n) is not null, it can be useful to have a method for showing this. You saw in Unit D2 that if (a_n) is a null sequence, then $(|a_n|)$ is also null, as are all subsequences of $(|a_n|)$. This leads to the following strategy.

Strategy D9

To show that $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent using the Non-null Test, check that the sequence (a_n) is not null by showing that $(|a_n|)$ has either

(a) a convergent subsequence with non-zero limit,

or

(b) a subsequence which tends to infinity.

Often when we apply Strategy D9 there is no need to consider a subsequence, because the *whole* sequence $(|a_n|)$ tends to a non-zero limit or to infinity, as in the following Worked Exercise.

Worked Exercise D31

Prove that each of the following series is divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n}$

Solution

(a) Let $a_n = (-1)^n$. Then $|a_n| = 1$.

We apply Strategy D9 to the whole sequence $|a_n|$.

So, since

$$\lim_{n\to\infty} |a_n| = 1 \neq 0,$$

the sequence (a_n) is not null. Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty}(-1)^n \quad \text{is divergent.}$$

(b) Let $a_n = 2^n/n$.

Here there is no need to consider $|a_n|$ as a_n is positive. The dominant term in the formula for a_n is in the numerator, so we use the Reciprocal Rule from Unit D2 to show that $a_n \to \infty$.

Then $(1/a_n) = (n/2^n)$ is a basic null sequence. So, by the Reciprocal Rule for sequences, $a_n \to \infty$ as $n \to \infty$. Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty} \frac{2^n}{n} \quad \text{is divergent.}$$

Exercise D45

Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{2n^2+1}$ is divergent.

Note that the converse of the Non-null Test is *false*. If the sequence (a_n) is null, then it is not necessarily true that the series $\sum_{n=1}^{\infty} a_n$ is convergent. For example, the sequence $(1/n)$ is null, but the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{is divergent.}$$

(We prove this rather surprising fact at the beginning of the next section.) So you can *never* use the Non-null Test to prove that a series is convergent.

2 Series with non-negative terms

In this section we restrict our attention to series $\sum_{n=1}^{\infty} a_n$ with *non-negative* terms. In other words, we assume that

$$a_n \geq 0, \quad \text{for } n = 1, 2, \ldots.$$

It follows that the partial sums of $\sum_{n=1}^{\infty} a_n$, given by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n, \end{aligned}$$

form an *increasing* sequence. This is illustrated in Figure 4 and in the sequence diagram in Figure 5.

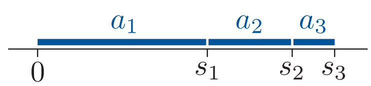


Figure 4 A series with non-negative terms

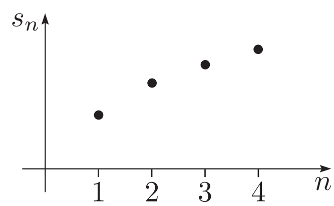


Figure 5 The partial sums of a series with non-negative terms

As in Section 1, we are interested in finding out whether a series with non-negative terms is convergent, even if we are unable to evaluate its sum or partial sums. The fact that the sequence (s_n) of partial sums is *increasing* helps us to determine whether (s_n) is convergent, since we can use the Monotone Convergence Theorem which you met in Unit D2. We restate this result below.

Theorem D29 Monotone Convergence Theorem

If the sequence (a_n) is either

- increasing and bounded above, or
- decreasing and bounded below,

then (a_n) is convergent.

Thus, if we can prove that the sequence (s_n) of partial sums of a series with non-negative terms is bounded above, then it follows from the Monotone Convergence Theorem that (s_n) is convergent, and hence that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

In this section you will meet several tests that can be used to check whether a series with non-negative terms is convergent and also several examples of such series.

2.1 Tests for convergence

We begin by studying two important examples of series with non-negative terms:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots .$$

Exercise D46

Use a calculator to find the first eight partial sums of each of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ (giving your answers to two decimal places), and plot your answers on a sequence diagram.

From the solution to Exercise D46, it appears that

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges.}$$

We now show that this is indeed the case. To do this, we look carefully at the partial sums of these two series to see whether the sequences of their partial sums converge or diverge.

We begin by considering $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. This series is often called the *harmonic* series, since its terms are proportional to the lengths of strings that produce harmonic tones in music.

The earliest recorded proof that the harmonic series is divergent is in a treatise dating from c.1350 by the French medieval philosopher Nicole Oresme (c.1320–1382).

Worked Exercise D32

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution

Notice that the sequence $(1/n)$ is null, so we cannot use the Non-null Test here. The key step in the proof is to find a subsequence of partial sums that tends to infinity. We do this by arranging the terms of the partial sums into groups.

Let s_n be the nth partial sum of the series. Then:

$$\begin{aligned} s_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{n} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\ &\qquad\qquad \cdots + \left(\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^k+2^k}\right) + \cdots . \end{aligned}$$

The kth bracket contains 2^k terms, each at least equal to $\dfrac{1}{2^k+2^k}$, so the sum of the terms in each bracket is at least equal to $\dfrac{2^k}{2^k+2^k} = \dfrac{1}{2}$. This fact enables us to use the Squeeze Rule for sequences which tend to infinity, which you met in Unit D2.

It follows that the subsequence (s_{2^k}) of partial sums is:

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + \frac{1}{2}, \\ s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \frac{1}{2}, \\ s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}, \\ &\vdots \\ s_{2^k} &> 1 + \underbrace{\frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}}_{k \text{ terms}} = 1 + \frac{1}{2}k. \end{aligned}$$

Since $1 + \frac{1}{2}k \to \infty$ as $k \to \infty$, it follows from the Squeeze Rule that $s_{2^k} \to \infty$.

We now use the Second Subsequence Rule from Unit D2, which says that if a sequence has a subsequence that tends to infinity, then the sequence is divergent.

Hence the sequence (s_n) is divergent by the Second Subsequence Rule.

It follows that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

We now consider the second series from Exercise D46.

Worked Exercise D33

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Solution

All the terms of the series are positive, so the sequence of partial sums is increasing. We show that the sequence of partial sums is bounded above and use the Monotone Convergence Theorem.

Let s_n be the nth partial sum of the series. Then, using the method of telescoping cancellation and the fact that

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$$

for all integers $k > 1$, we have

$$\begin{aligned}
s_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots + \frac{1}{n^2} \\
&< 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{(n-1) \times n} \\
&= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 2 - \frac{1}{n} \\
&< 2.
\end{aligned}$$

It follows that (s_n) is both increasing and bounded above (by 2, for example) so that, by the Monotone Convergence Theorem, (s_n) is convergent. So the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

In fact, remarkably, it can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

However, a proof of this (which depends on use of trigonometric series or of complex analysis) goes beyond the scope of this module.

The problem of finding the exact sum of the series $\sum_{n=1}^{\infty}(1/n^2)$ is known as the Basel problem. The problem was first posed by the Italian mathematician Pietro Mengoli (1626–1686) in 1644. After withstanding the attack of many leading mathematicians, including several members of the Bernoulli family, it was solved by the young Leonhard Euler (1707–1783) in 1734, bringing him immediate fame. The problem is named after the city of Basel in Switzerland, the hometown of Euler and of the Bernoulli family.

Just as with sequences, there are some general results that enable us to deduce the convergence or divergence of a given series from the known convergence or divergence of another series. Our next result is of this type.

Theorem D30 Comparison Test

(a) If

$$0 \le a_n \le b_n, \quad \text{for } n = 1, 2, \ldots, \tag{2}$$

and $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If

$$0 \le b_n \le a_n, \quad \text{for } n = 1, 2, \ldots, \tag{3}$$

and $\displaystyle\sum_{n=1}^{\infty} b_n$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Proof (a) Assume that inequality (2) holds. Then the nth partial sums

$$s_n = a_1 + a_2 + \cdots + a_n, \quad n = 1, 2, \ldots,$$

and

$$t_n = b_1 + b_2 + \cdots + b_n, \quad n = 1, 2, \ldots,$$

satisfy

$$s_n \le t_n, \quad \text{for } n = 1, 2, \ldots.$$

We also know that $\sum_{n=1}^{\infty} b_n$ is convergent, so the increasing sequence (t_n) is convergent with limit t, say. Hence

$$s_n \le t_n \le t, \quad \text{for } n = 1, 2, \ldots,$$

so the increasing sequence (s_n) is bounded above by t. By the Monotone Convergence Theorem, (s_n) is also convergent, so $\sum_{n=1}^{\infty} a_n$ is a convergent series.

(b) Now assume that inequality (3) holds. Notice that the statement

$$\text{if } \sum_{n=1}^{\infty} b_n \text{ is divergent, then } \sum_{n=1}^{\infty} a_n \text{ is divergent}$$

is equivalent to the statement

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \sum_{n=1}^{\infty} b_n \text{ is convergent.}$$

The second statement is the *contrapositive* of the first statement and so is equivalent to it, as you saw in Unit A3 *Mathematical language and proof*.

But the second statement follows from part (a) with the roles of a_n and b_n interchanged, so this proves part (b). ■

Remarks

1. Informally, in part (a) we say that $\sum_{n=1}^{\infty} a_n$ 'is dominated by' $\sum_{n=1}^{\infty} b_n$; and in part (b) that $\sum_{n=1}^{\infty} a_n$ 'dominates' $\sum_{n=1}^{\infty} b_n$.
2. In the proof of part (a), if we apply the Limit Inequality Rule from Unit D2 to the inequality

   $$s_n \leq t_n, \quad \text{for } n = 1, 2, \ldots,$$

   then we can in fact deduce that $\displaystyle\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.
3. When using the Comparison Test, it is sufficient to show that the necessary inequalities in parts (a) and (b) hold *eventually*; that is, that $0 \leq a_n \leq b_n$ or $0 \leq b_n \leq a_n$ for all $n > N$, for some number N. (You met this idea of a property holding *eventually* in Unit D2.)

In applications, we use the Comparison Test in the following way – sometimes informally called a 'guess then check' approach.

Strategy D10

(a) To show that a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms is *convergent* using the Comparison Test, do the following.

1. Guess that $\sum_{n=1}^{\infty} a_n$ is dominated by a convergent series $\sum_{n=1}^{\infty} b_n$.
2. Check that $0 \leq a_n \leq b_n$, for $n = 1, 2, \ldots$.

(b) To show that a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms is *divergent* using the Comparison Test, do the following.

1. Guess that $\sum_{n=1}^{\infty} a_n$ dominates a divergent series $\sum_{n=1}^{\infty} b_n$.
2. Check that $0 \leq b_n \leq a_n$, for $n = 1, 2, \ldots$.

In either case, the first step is to find a suitable series $\sum_{n=1}^{\infty} b_n$ to compare with the series $\sum_{n=1}^{\infty} a_n$ that we are investigating. To do this, we choose a series whose nth term b_n seems likely to be greater than or equal to a_n (less than or equal to a_n) and which we know converges (diverges). Carrying out the check in step 2 of Strategy D10 will then show whether or not our guess of $\sum_{n=1}^{\infty} b_n$ was suitable. This is illustrated in the next two worked exercises.

Worked Exercise D34

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

Solution

We need to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is dominated by a series which we know to be convergent.

We guess that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is dominated by $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. Indeed, since

$$n^3 \geq n^2 \geq 0, \quad \text{for } n = 1, 2, \ldots,$$

we have

$$0 \leq \frac{1}{n^3} \leq \frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (by Worked Exercise D33), we deduce from part (a) of the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent.

There is no simple way of writing the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. This value is now known as Apéry's constant after the French mathematician Roger Apéry who in 1978 proved that it is irrational. Apéry's constant arises naturally in a number of physical problems; for example, in the quantification of black body energy radiation.

Worked Exercise D35

Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Solution

We guess that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ dominates $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Indeed, since

$$n \geq \sqrt{n} \geq 0, \quad \text{for } n = 1, 2, \ldots,$$

we have

$$0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}, \quad \text{for } n = 1, 2, \ldots.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (by Worked Exercise D32), we deduce from part (b) of the Comparison Test that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Next we consider the series

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+1}. \qquad (4)$$

This seems likely to be divergent, because its terms are somewhat similar to those of $\sum_{n=1}^{\infty} (1/\sqrt{n})$, which we showed to be divergent in Worked Exercise D35. But how can we prove this?

We cannot use the Comparison Test for these two series because the inequality in step 2 of Strategy D10(b) does not hold. We *could* use the Comparison Test to compare series (4) with $\sum_{n=1}^{\infty}(1/n)$, in which case we would find that the inequality in step 2 holds eventually. However, the following useful result enables us to deduce the divergence of series (4) directly from the divergence of $\sum_{n=1}^{\infty} (1/\sqrt{n})$.

Theorem D31 Limit Comparison Test

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ have positive terms and that

$$\frac{a_n}{b_n} \to L \quad \text{as } n \to \infty,$$

where $L \neq 0$.

(a) If $\displaystyle\sum_{n=1}^{\infty} b_n$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof (a) We know that the sequence (a_n/b_n) is convergent, so it must be bounded by Theorem D14 of Unit D2. Thus there is a positive constant K such that

$$\frac{a_n}{b_n} \leq K, \quad \text{for } n = 1, 2, \ldots,$$

so

$$a_n \leq K b_n, \quad \text{for } n = 1, 2, \ldots.$$

We also know that $\sum_{n=1}^{\infty} b_n$ is convergent, so $\sum_{n=1}^{\infty} K b_n$ is convergent, by the Multiple Rule. Hence, by part (a) of the Comparison Test, $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) As we saw in the proof of part (b) of the Comparison Test, the statement

$$\text{if } \sum_{n=1}^{\infty} b_n \text{ is divergent, then } \sum_{n=1}^{\infty} a_n \text{ is divergent}$$

is the contrapositive of the statement

$$\text{if } \sum_{n=1}^{\infty} a_n \text{ is convergent, then } \sum_{n=1}^{\infty} b_n \text{ is convergent,}$$

so these two statements are equivalent. Now since $L \neq 0$, we have

$$\frac{b_n}{a_n} \to \frac{1}{L} \text{ as } n \to \infty,$$

by the Quotient Rule for sequences from Unit D2. Thus the second statement follows from part (a) with the roles of a_n and b_n interchanged, which proves part (b). ■

Remarks

1. The hypothesis $a_n/b_n \to L$ can be interpreted as saying that 'a_n behaves rather like b_n for large n'.
2. The assumption that $L \neq 0$ is not needed in the proof of part (a), but it is essential in the proof of part (b).

As we did with the Comparison Test, we can formulate a convenient way to use the Limit Comparison Test via a 'guess then check' approach.

Strategy D11

(a) To show that a series $\sum_{n=1}^{\infty} a_n$ of positive terms is *convergent* using the Limit Comparison Test, do the following.

1. Guess that $\sum_{n=1}^{\infty} a_n$ behaves like a comparable convergent series $\sum_{n=1}^{\infty} b_n$ of positive terms.
2. Check that $\dfrac{a_n}{b_n} \to L \neq 0$ as $n \to \infty$ and deduce that $\sum_{n=1}^{\infty} a_n$ converges.

(b) To show that a series $\sum_{n=1}^{\infty} a_n$ of positive terms is *divergent* using the Limit Comparison Test, do the following.

1. Guess that $\sum_{n=1}^{\infty} a_n$ behaves like a comparable divergent series $\sum_{n=1}^{\infty} b_n$ of positive terms.
2. Check that $\dfrac{a_n}{b_n} \to L \neq 0$ as $n \to \infty$ and deduce that $\sum_{n=1}^{\infty} a_n$ diverges.

In either case, the first step is to find a suitable series $\sum_{n=1}^{\infty} b_n$ of positive terms to compare with the series $\sum_{n=1}^{\infty} a_n$ that we are investigating. To do this, we choose a series that we know to be convergent or divergent, and whose nth term b_n seems likely to behave in a similar way to a_n for large values of n. Carrying out the check in step 2 of Strategy D11 will then show whether or not our guess of $\sum_{n=1}^{\infty} b_n$ was suitable. This is illustrated in the next worked exercise.

Worked Exercise D36

Determine whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{3n^4-n}$

Solution

(a) Following the discussion after Worked Exercise D35, we guess that terms of this series behave like $1/\sqrt{n}$ for large n. We know that $\sum_{n=1}^{\infty} (1/\sqrt{n})$ is divergent.

We use the Limit Comparison Test with

$$a_n = \frac{1}{2\sqrt{n}+1}, \quad \text{for } n = 1, 2, \ldots$$

and

$$b_n = \frac{1}{\sqrt{n}}, \quad \text{for } n = 1, 2, \ldots.$$

Both a_n and b_n are positive, and

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{1}{2\sqrt{n}+1}\bigg/\frac{1}{\sqrt{n}}\\ &= \frac{\sqrt{n}}{2\sqrt{n}+1}\\ &= \frac{1}{2+1/\sqrt{n}} \to \frac{1}{2} \neq 0.\end{aligned}$$

This follows from the Combination Rules for sequences since $(1/\sqrt{n})$ is a basic null sequence.

Since $\sum_{n=1}^{\infty}(1/\sqrt{n})$ is divergent, it follows from part (b) of the Limit Comparison Test that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}+1}$ is divergent.

(b) For large n, the expression $\dfrac{n+5}{3n^4-n}$ is approximately $\dfrac{n}{3n^4}=\dfrac{1}{3n^3}$, so we guess that the terms of this series behave rather like $1/n^3$ for large n. We know that the series $\sum_{n=1}^{\infty}(1/n^3)$ is convergent.

We use the Limit Comparison Test with

$$a_n = \frac{n+5}{3n^4-n}, \quad \text{for } n = 1, 2, \ldots$$

and

$$b_n = \frac{1}{n^3}, \quad \text{for } n = 1, 2, \ldots.$$

Both a_n and b_n are positive, and

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{n+5}{3n^4-n} \times \frac{n^3}{1}\\ &= \frac{n^4+5n^3}{3n^4-n}\\ &= \frac{1+5/n}{3-1/n^3} \to \frac{1}{3} \neq 0.\end{aligned}$$

This follows from the Combination Rules for sequences since $(1/n)$ and $(1/n^3)$ are basic null sequences.

Since $\sum_{n=1}^{\infty}(1/n^3)$ is convergent, it follows from part (a) of the Limit Comparison Test that $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{3n^4-n}$ is convergent.

In the next exercise you can practise using the Comparison Test and the Limit Comparison Test. If you can see a direct comparison with a series which you know to be convergent or divergent, then you can use the Comparison Test. If the terms of the series just behave like the terms of a known series for large values of n, then you should use the Limit Comparison Test.

Exercise D47

Use the Comparison Test or the Limit Comparison Test to determine whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+4}{2n^3-n+1}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(2n)}{n^3}$

Our next test for convergence is motivated in part by the properties of geometric series. Recall that the geometric series

$$a + ar + ar^2 + \cdots = \sum_{n=1}^{\infty} ar^n, \quad \text{where } a \neq 0,$$

is convergent if $|r| < 1$ but divergent if $|r| \geq 1$.

Theorem D32 Ratio Test

Suppose that $\displaystyle\sum_{n=1}^{\infty} a_n$ has positive terms and that $\dfrac{a_{n+1}}{a_n} \to l$ as $n \to \infty$.

(a) If $0 \leq l < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $l > 1$ or $l = \infty$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

Proof (a) We know that $0 \leq l < 1$, so we can choose $\varepsilon > 0$ such that

$$l + \varepsilon < 1.$$

(For example, we can take $\varepsilon = \frac{1}{2}(1-l)$.) Let $r = l + \varepsilon$. Since $r > l$, there is a positive integer N such that

$$\frac{a_{n+1}}{a_n} \leq r, \quad \text{for all } n \geq N.$$

This is illustrated in Figure 6. Thus, for $n \geq N$, we have

$$\frac{a_n}{a_N} = \left(\frac{a_n}{a_{n-1}}\right)\left(\frac{a_{n-1}}{a_{n-2}}\right)\cdots\left(\frac{a_{N+1}}{a_N}\right) \leq r^{n-N},$$

since each of the expressions in brackets is at most r. Hence

$$a_n \leq a_N r^{n-N}, \quad \text{for } n \geq N. \tag{5}$$

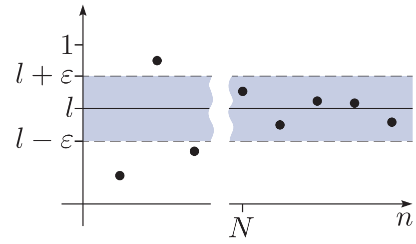


Figure 6 The sequence diagram for (a_{n+1}/a_n) if $0 \leq l < 1$

Now

$$\sum_{n=N}^{\infty} a_N r^{n-N} = a_N + a_N r + a_N r^2 + \cdots$$

is a geometric series with first term a_N and common ratio r. Since $0 < r < 1$, this series is convergent. Thus, by inequality (5) and the Comparison Test, $\sum_{n=1}^{\infty} a_n$ is also convergent, as required.

(b) Since

$$\frac{a_{n+1}}{a_n} \to \infty \quad \text{or} \quad \frac{a_{n+1}}{a_n} \to l,$$

where $l > 1$, there is a positive integer N such that

$$\frac{a_{n+1}}{a_n} \geq 1, \quad \text{for all } n \geq N,$$

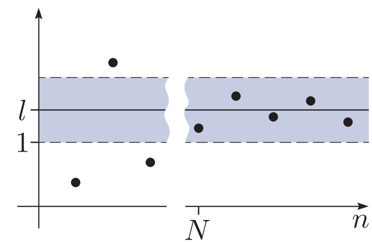


Figure 7 The sequence diagram for (a_{n+1}/a_n) if $l > 1$

as shown in Figure 7. Thus, for $n \geq N$, we have

$$\frac{a_n}{a_N} = \left(\frac{a_n}{a_{n-1}}\right)\left(\frac{a_{n-1}}{a_{n-2}}\right)\cdots\left(\frac{a_{N+1}}{a_N}\right) \geq 1,$$

since each of the expressions in brackets is at least 1. Hence

$$a_n \geq a_N > 0, \quad \text{for } n \geq N,$$

so (a_n) cannot be a null sequence. It follows, by the Non-null Test, that $\sum_{n=1}^{\infty} a_n$ is divergent. ■

Remarks

1. With the Ratio Test, we concentrate on the series $\sum_{n=1}^{\infty} a_n$ itself and do not need to compare it with some other series $\sum_{n=1}^{\infty} b_n$.
2. If $l = 1$, the Ratio Test gives us no information on whether the series converges. (That is, the Ratio Test is *inconclusive* if $l = 1$.)

   For example, if $a_n = \dfrac{1}{n}$ then

   $$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} = \frac{1}{1+1/n} \to 1 \text{ as } n \to \infty$$

   and we have seen (in Worked Exercise D32) that the series $\sum_{n=1}^{\infty}(1/n)$ diverges.

On the other hand, if $a_n = \dfrac{1}{n^2}$ then

$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \frac{1}{1+2/n+1/n^2} \to 1 \text{ as } n \to \infty,$$

but as we have seen (in Worked Exercise D33), the series $\sum_{n=1}^{\infty}(1/n^2)$ converges.

3. When using the Ratio Test, we obtain a_{n+1} by replacing each instance of n by $n+1$ in the formula for a_n.

Jean le Rond d'Alembert

The Ratio Test was first published by the French mathematician Jean le Rond d'Alembert (1717–1783) in 1768. D'Alembert's interest in mechanics led him to take a Newtonian view of the foundations of the calculus, and he was among the first to regard the method of limits as fundamental. He was the editor of the mathematical and scientific articles in the *Encylopédie* – a remarkable series of volumes which constitutes one of the major documents of the Enlightenment – and wrote many of the articles himself, including the ones on *Differéntiel* and *Limite*.

Worked Exercise D37

Use the Ratio Test to determine whether the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$

Solution

(a) Let

$$a_n = \frac{n}{2^n}, \quad \text{for } n = 1, 2, \ldots.$$

Then a_n is positive, and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{n+1}{2^{n+1}} \times \frac{2^n}{n} \\ &= \frac{n+1}{2n} \\ &= \frac{1+1/n}{2} \to \frac{1}{2} \text{ as } n \to \infty.\end{aligned}$$

Since $0 < \frac{1}{2} < 1$, it follows from the Ratio Test that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent.

(b) Let

$$a_n = \frac{10^n}{n!}, \quad \text{for } n = 1, 2, \ldots.$$

Then a_n is positive, and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{10^{n+1}}{(n+1)!} \times \frac{n!}{10^n} \\ &= \frac{10}{n+1} \to 0 \text{ as } n \to \infty.\end{aligned}$$

It follows from the Ratio Test that $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{n!}$ is convergent.

Exercise D48

Use the Ratio Test to determine whether the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}$

Hint: In part (c) you need to use the fact (from Unit D2) that $(1+1/n)^n \to e$ as $n \to \infty$.

2.2 Basic series

When studying sequences in Unit D2, we made great use of a library of basic sequences. You will now see that there is also a library of basic series whose convergence or divergence is known. We can determine the convergence or divergence of a large number of other series from these basic series by using our tests.

Theorem D33 Basic series

The following series are convergent:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, for $p \geq 2$

(b) $\displaystyle\sum_{n=1}^{\infty} c^n$, for $0 \leq c < 1$

(c) $\displaystyle\sum_{n=1}^{\infty} n^p c^n$, for $p > 0$, $0 \leq c < 1$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{c^n}{n!}$, for $c \geq 0$.

The following series is divergent:

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$, for $0 < p \leq 1$.

Proof (a) This series is convergent, by the Comparison Test, since if $p \geq 2$, then

$$\frac{1}{n^p} \leq \frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots,$$

and the series $\sum_{n=1}^{\infty}(1/n^2)$ is convergent.

(In Unit F3 *Integration* we will prove that $\sum_{n=1}^{\infty}(1/n^p)$ is convergent, for all $p > 1$.)

(b) The series $\sum_{n=1}^{\infty} c^n$ is a geometric series with common ratio c, so it converges if $0 \leq c < 1$.

(c) Let

$$a_n = n^p c^n, \quad n = 1, 2, \ldots.$$

Then put $b = \sqrt{c}$ and express a_n as

$$a_n = n^p (b \times b)^n = (n^p b^n) b^n, \quad \text{for } n = 1, 2, \ldots. \tag{6}$$

Now $0 \leq b < 1$, so $(n^p b^n)$ is a basic null sequence. Hence, for some positive integer N, we have

$$n^p b^n < 1, \quad \text{for } n > N,$$

and thus, by equation (6),

$$a_n < b^n, \quad \text{for } n > N.$$

But $\sum_{n=1}^{\infty} b^n$ is a convergent geometric series, so $\sum_{n=1}^{\infty} a_n$ is convergent by the Comparison Test.

(d) Let

$$a_n = \frac{c^n}{n!}, \quad n = 1, 2, \ldots.$$

If $c = 0$, then the series is clearly convergent. For $c \neq 0$,

$$\frac{a_{n+1}}{a_n} = \frac{c^{n+1}}{(n+1)!} \bigg/ \frac{c^n}{n!} = \frac{c^{n+1}}{(n+1)!} \times \frac{n!}{c^n} = \frac{c}{n+1}.$$

Thus

$$\frac{a_{n+1}}{a_n} \to 0 \quad \text{as } n \to \infty$$

and we deduce from the Ratio Test that $\displaystyle\sum_{n=1}^{\infty} \frac{c^n}{n!}$ is convergent.

(e) This series is divergent by the Comparison Test, since if $p \leq 1$, then

$$\frac{1}{n^p} \geq \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots$$

and $\sum_{n=1}^{\infty}(1/n)$ is divergent. ■

Exercise D49

Verify that the following are all basic series, stating their type according to the list in Theorem D33 and giving the values of any parameters p and c. Hence determine which of the series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$

3 Series with positive and negative terms

The study of series $\sum a_n$ with $a_n \geq 0$ for all values of n is relatively straightforward because the sequence (s_n) of partial sums is increasing. Similarly, if $a_n \leq 0$ for all values of n, then (s_n) is decreasing.

It is harder to determine the behaviour of a series with both positive and negative terms because (s_n) is neither increasing nor decreasing. However, if the sequence (a_n) contains only finitely many negative terms, then the sequence (s_n) is *eventually* increasing, and we can apply the results of Section 2. Similarly, if (a_n) contains only finitely many positive terms, then the sequence (s_n) is *eventually* decreasing, and we can again apply the results of Section 2 to the series $\sum_{n=1}^{\infty}(-a_n)$. For example, the convergence of the series

$$1 + 2 + 3 - \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} - \cdots$$

follows from that of $\sum_{n=1}^{\infty}(1/n^2)$, by the Multiple Rule with $\lambda = -1$.

In this section we look at series such as

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

and

$$1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots,$$

which contain infinitely many terms of either sign. For such series the partial sums alternately increase and decrease infinitely often (so the snake discussed in Subsection 1.1 grows and shrinks infinitely often!). We give two methods which can often be used to prove that such series are convergent.

3.1 Absolute convergence

Suppose that we want to determine the behaviour of the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots . \qquad (7)$$

We know that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \qquad (8)$$

is convergent. Does this imply that series (7) is also convergent? In fact it does, as we now prove.

Consider the two related series

$$1 + 0 + \frac{1}{3^2} + 0 + \frac{1}{5^2} + 0 + \cdots$$

and

$$0 + \frac{1}{2^2} + 0 + \frac{1}{4^2} + 0 + \frac{1}{6^2} + \cdots .$$

Each of these series contains only non-negative terms and is dominated by series (8), so they are both convergent, by the Comparison Test. Applying the Sum Rule for series, and the Multiple Rule for series with $\lambda = -1$, we deduce that the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$$

is convergent.

The type of argument just given is the basis for a concept called *absolute convergence*, which we now define.

Definition

The series $\displaystyle\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** if $\displaystyle\sum_{n=1}^{\infty} |a_n|$ is convergent.

If the terms a_n are all non-negative, then *absolute convergence* and *convergence* have the same meaning.

The series (7) is absolutely convergent because the series $\sum_{n=1}^{\infty}(1/n^2)$ is convergent. However, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \qquad (9)$$

is not absolutely convergent because the series $\sum_{n=1}^{\infty}(1/n)$ is divergent.

As the name suggests, every absolutely convergent series is convergent.

Theorem D34 Absolute Convergence Test

If $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

Proof We know that $\sum_{n=1}^{\infty} |a_n|$ is convergent, and we want to prove that $\sum_{n=1}^{\infty} a_n$ is convergent.

To do this, we define two new series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$, where

$$b_n = \begin{cases} a_n, & \text{if } a_n \geq 0, \\ 0, & \text{if } a_n < 0, \end{cases} \qquad c_n = \begin{cases} 0, & \text{if } a_n \geq 0, \\ -a_n, & \text{if } a_n < 0. \end{cases}$$

Both the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ have non-negative terms, and

$$b_n \leq |a_n|, \quad \text{for } n = 1, 2, \ldots, \tag{10}$$

and

$$c_n \leq |a_n|, \quad \text{for } n = 1, 2, \ldots. \tag{11}$$

Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, we deduce that $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ are convergent, by the Comparison Test. Thus

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - c_n) = \sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} c_n \tag{12}$$

is convergent, by the Combination Rules for series. ■

Because $\sum_{n=1}^{\infty}(1/n^2)$ is convergent, it follows from the Absolute Convergence Test that series (7) is convergent, as we have already seen. Indeed, however we distribute the plus and minus signs amongst the terms of the sequence $(1/n^2)$, the resulting series is convergent.

However, the Absolute Convergence Test tells us nothing about the behaviour of series (9), nor about similar series such as

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \cdots. \tag{13}$$

The series $\sum_{n=1}^{\infty}(1/n)$ is divergent, so these two series are not absolutely convergent. You will see later how to use other methods to determine whether series (9) and (13) are convergent.

Worked Exercise D38

Prove that the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

Solution

(a) Let

$$a_n = \frac{(-1)^{n+1}}{n^3}, \quad \text{for } n = 1, 2, \dots.$$

Then

$$|a_n| = \frac{1}{n^3}, \quad \text{for } n = 1, 2, \dots.$$

We know that $\sum_{n=1}^{\infty}(1/n^3)$ is convergent, so by the Absolute Convergence Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \text{ is convergent.}$$

(b) Let

$$a_n = \frac{\cos n}{2^n}, \quad \text{for } n = 1, 2, \dots.$$

In this case we use the fact that $|\cos n| \leq 1$ to get an upper bound for the size of $|a_n|$. We can then use the Comparison Test to show that $\sum |a_n|$ is convergent.

Then

$$|a_n| \leq \frac{1}{2^n}, \quad \text{for } n = 1, 2, \dots,$$

because $|\cos n| \leq 1$, for $n = 1, 2, \dots$.

Since $\sum_{n=1}^{\infty}(1/2^n)$ is a basic convergent series, it follows from the Comparison Test that $\sum_{n=1}^{\infty} |a_n|$ is convergent. Hence, by the Absolute Convergence Test,

$$\sum_{n=1}^{\infty} \frac{\cos n}{2^n} \text{ is convergent.}$$

Exercise D50

Prove that the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^3 + 1}$ (b) $1 + \dfrac{1}{2} - \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16} - \dfrac{1}{32} + \cdots$

The Absolute Convergence Test states that if the series $\sum |a_n|$ is convergent, then $\sum a_n$ is also convergent. The following result relates the sums of these two convergent series. This result generalises the Triangle Inequality for n real numbers,

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|,$$

that is,

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|,$$

which you met in Subsection 3.1 of Unit D1 *Numbers*.

Theorem D35 Triangle Inequality (infinite form)

If $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Proof We use the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$, introduced in the proof of the Absolute Convergence Test. Since the numbers b_n and c_n are all non-negative, we obtain the following inequalities from equation (12):

$$-\sum_{n=1}^{\infty} c_n \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n.$$

Thus, by inequalities (10) and (11), we deduce that

$$-\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n|,$$

which gives the required inequality

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

■

Exercise D51

Show that the series

$$\frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} - \frac{1}{64} + \cdots$$

is convergent, and that its sum lies in the interval $[-1, 1]$.

(In this series the signs of the terms are $+$, $-$, $-$, repeated infinitely often. You do not need to find the sum of the series.)

3.2 Alternating Test

Suppose that we want to determine the behaviour of the following infinite series, in which the terms have alternating signs:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots . \tag{14}$$

(You met this series earlier as series (9).) The Absolute Convergence Test does not help us with this series because $\sum_{n=1}^{\infty}(1/n)$ is divergent. In fact, series (14) is convergent. To see why, we first calculate some of the partial sums and plot them on a sequence diagram:

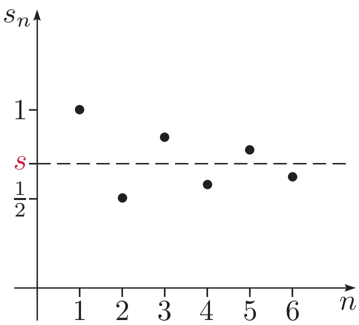


Figure 8 The sequence diagram for (s_n)

$$\begin{aligned}
s_1 &= 1,\\
s_2 &= 1 - \frac{1}{2} = 0.5,\\
s_3 &= 1 - \frac{1}{2} + \frac{1}{3} = 0.8\overline{3},\\
s_4 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = 0.58\overline{3},\\
s_5 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} = 0.78\overline{3},\\
s_6 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} = 0.61\overline{6}.
\end{aligned}$$

The sequence diagram is shown in Figure 8. This suggests that the odd subsequence (s_{2k-1}) is decreasing:

$$s_1 \geq s_3 \geq s_5 \geq \cdots \geq s_{2k-1} \geq \cdots ,$$

and that the even subsequence (s_{2k}) is increasing:

$$s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2k} \leq \cdots .$$

Also, the terms of (s_{2k-1}) all exceed the terms of (s_{2k}), and both subsequences appear to converge to a common limit s, which lies between the odd and even partial sums.

To prove this, we write the even partial sum s_{2k} as

$$s_{2k} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right).$$

All the expressions in brackets are positive, so the subsequence (s_{2k}) is increasing.

We can also write

$$s_{2k} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2k-2} - \frac{1}{2k-1}\right) - \frac{1}{2k}.$$

Again, all the expressions in brackets are positive, so (s_{2k}) is bounded above by 1.

Hence, by the Monotone Convergence Theorem,

$$\lim_{k\to\infty} s_{2k} = s,$$

for some s. Since

$$s_{2k} = s_{2k-1} - \frac{1}{2k}, \quad \text{for } k = 1, 2, \ldots,$$

and the sequence $(1/2k)$ is null, we deduce that

$$\lim_{k\to\infty} s_{2k-1} = \lim_{k\to\infty} \left(s_{2k} + \frac{1}{2k}\right) = s,$$

by the Sum Rule for sequences. Thus the odd and even subsequences of (s_n) both tend to the same limit s, so $s_n \to s$ as $n \to \infty$, by Theorem D21 in Unit D2. Hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent, with sum s.

(In fact, $s = \log_e 2 \approx 0.69$ but we do not show this here.)

The same method can be used to prove the following general result which is also called the Leibniz Test.

Theorem D36 Alternating Test

Let

$$a_n = (-1)^{n+1} b_n, \quad n = 1, 2, \ldots,$$

where (b_n) is a decreasing null sequence with positive terms. Then

$$\sum_{n=1}^{\infty} a_n = b_1 - b_2 + b_3 - b_4 + \cdots \text{ is convergent.}$$

Proof We can write any even partial sum s_{2k} as

$$s_{2k} = (b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{2k-1} - b_{2k}).$$

Since the sequence (b_n) is decreasing, all the expressions in brackets are non-negative, so the even subsequence (s_{2k}) is increasing.

We can also write

$$s_{2k} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2k-2} - b_{2k-1}) - b_{2k}.$$

Again, all the expressions in brackets are non-negative, so the subsequence (s_{2k}) is bounded above by b_1.

Hence, by the Monotone Convergence Theorem,

$$\lim_{k\to\infty} s_{2k} = s,$$

for some s. Since

$$s_{2k} = s_{2k-1} - b_{2k}, \quad \text{for } k = 1, 2, \ldots,$$

and the sequence (b_n) is null, we deduce that

$$\lim_{k\to\infty} s_{2k-1} = \lim_{k\to\infty} (s_{2k} + b_{2k}) = s,$$

by the Sum Rule for sequences. Thus the odd and even subsequences of (s_n) both tend to the same limit s. Hence, by Theorem D21 in Unit D2, s_n tends to s, so $\sum_{n=1}^{\infty} a_n$ is convergent, with sum s. ■

When you apply the Alternating Test there are a number of conditions to check. We now describe these in the form of a strategy.

Strategy D12

To prove that $\sum_{n=1}^{\infty} a_n$ is convergent using the Alternating Test, check that

$$a_n = (-1)^{n+1} b_n, \quad n = 1, 2, \ldots,$$

where

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. (b_n) is a null sequence
3. (b_n) is decreasing.

When you use this strategy and are checking that the sequence (b_n) is null, you may find it helpful to use the techniques you met in Unit D2, including the list of basic null sequences.

Here are some examples of the use of Strategy D12.

Worked Exercise D39

Prove that the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n^2 - 1}$

Solution

(a) Let

$$a_n = \frac{(-1)^{n+1}}{\sqrt{n}}, \quad \text{for } n = 1, 2, \ldots.$$

Then $a_n = (-1)^{n+1} b_n$, where

$$b_n = 1/\sqrt{n}, \quad \text{for } n = 1, 2, \ldots.$$

Now

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. (b_n) is a basic null sequence
3. (b_n) is decreasing, because $(1/b_n) = (\sqrt{n})$ is increasing.

Hence, by the Alternating Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \text{ is convergent.}$$

(b) Let

$$a_n = \frac{(-1)^{n+1} n}{2n^2 - 1}, \quad \text{for } n = 1, 2, \ldots.$$

Then $a_n = (-1)^{n+1} b_n$, where

$$b_n = \frac{n}{2n^2 - 1}, \quad \text{for } n = 1, 2, \ldots.$$

Now

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. since
$$b_n = \frac{1/n}{2 - 1/n^2} \to 0 \quad \text{as } n \to \infty,$$
(b_n) is null
3. (b_n) is decreasing, because
$$\left(\frac{1}{b_n}\right) = \left(\frac{2n^2 - 1}{n}\right) = \left(2n - \frac{1}{n}\right)$$
is increasing.

Hence, by the Alternating Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n^2 - 1} \text{ is convergent.}$$

Exercise D52

Determine which of the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/3}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + n^{1/2}}$ (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+2}$

3.3 General strategy

We now give a strategy for applying the tests for convergence (or divergence) of a series. For each test, the strategy briefly indicates the circumstances under which that test can be used.

Strategy D13

To determine whether a series $\sum a_n$ is convergent or divergent, do the following.

1. If you think that the sequence of terms (a_n) is non-null, then try the **Non-null Test**.
2. If $\sum a_n$ has non-negative terms, then try one of these tests.

   Basic series Is $\sum a_n$ a basic series, or a combination of these?

   Comparison Test Is $a_n \leq b_n$, where $\sum b_n$ is convergent, or $a_n \geq b_n \geq 0$, where $\sum b_n$ is divergent?

   Limit Comparison Test Does a_n behave like b_n for large n (that is, does $a_n/b_n \to L \neq 0$), where $\sum b_n$ is a series that you know converges or diverges?

   Ratio Test Does $a_{n+1}/a_n \to l \neq 1$?
3. If $\sum a_n$ has infinitely many positive and negative terms, then try one of these tests.

   Absolute Convergence Test Is $\sum |a_n|$ convergent? (Use step 2.)

   Alternating Test Is $a_n = (-1)^{n+1} b_n$, where (b_n) is non-negative, null and decreasing?

Remarks

1. When applying these tests, you do not need to try to prove that the sequence (a_n) is null (though in the case of the Alternating Test, you do need to show that the sequence (b_n) is null).
2. If the terms a_n of the series $\sum a_n$ are non-positive, apply step 2 of the strategy to the series $\sum(-a_n)$ and then use the Multiple Rule with $\lambda = -1$.
3. If none of steps 1–3 of the strategy gives a result, then you could try using first principles by working directly with the sequence (s_n) of partial sums.

4. The following suggestions may also be helpful.
   - If a_n is positive and includes $n!$ or c^n, then consider the Ratio Test.
   - If a_n is positive and has dominant term n^p, then consider the Comparison Test or the Limit Comparison Test.
   - If a_n includes a sine or cosine term, then use the fact that this term is bounded and consider the Comparison Test and the Absolute Convergence Test.

Exercise D53

Determine which of the following series are convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5n+2^n}{3^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2n^3-1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\log(n+1)}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n^2}{n^2+1}$ (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^3+5}$ (f) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^6}$

4 The exponential function

In this section you will see how e^x can be represented as an infinite series of powers of x. This representation is then used to prove that the number e is irrational, and also that $e^x e^y = e^{x+y}$ for any real numbers x and y. This section is not assessed and is for reading only.

4.1 Definition of e^x

In Subsection 5.3 of Unit D2 we defined $e = 2.718\,28\ldots$ to be the limit

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

We also stated that if $x > 0$, then

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n,$$

We can now use infinite series to give an alternative definition of e^x.

Figure 9 shows graphs of the first four partial sum functions

$$s_0(x) = 1, \quad s_1(x) = 1 + x, \quad s_2(x) = 1 + x + \frac{x^2}{2!}, \quad \ldots \quad ,$$

of the following infinite series of powers of x:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{15}$$

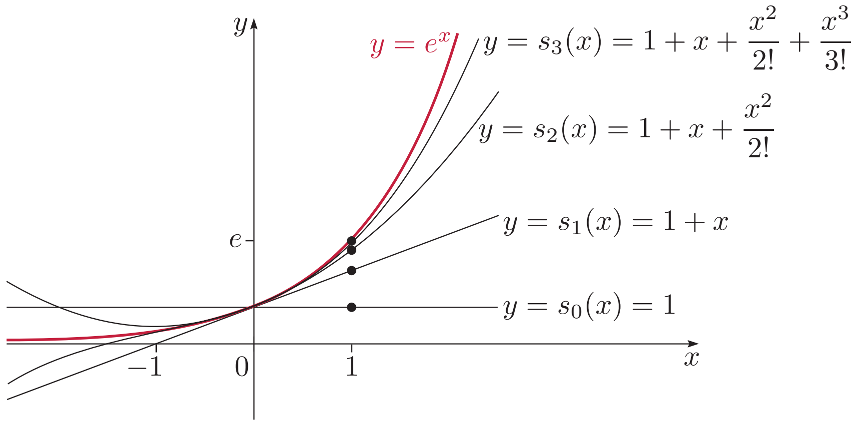


Figure 9 The partial sum functions of $\sum_{n=0}^{\infty}(x^n/n!)$

We know that series (15) is convergent for all real numbers x since it is a basic convergent series of type (a). As the sum of the series depends on x, it defines a function of x. If you test different values, then you will find that this sum function appears to be e^x. In particular, when $x = 1$, the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

is approximately $2.718\ldots$.

It can be proved that series (15) does converge to e^x for all $x \in \mathbb{R}$ and we give this result, for $x \geq 0$, as our next theorem. If you are short of time, then skim read the proof, noting the main points.

Theorem D37

If $x \geq 0$, then

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Proof We give the proof only for the case $x = 1$; the general case is similar. We have to show that

$$\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

The nth partial sum of this convergent series is

$$s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}, \quad \text{so} \quad \lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

Now, by the Binomial Theorem, we have

$$\left(1 + \frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)^n. \tag{16}$$

The general term in this expansion is of the form

$$\begin{aligned} &\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k \\ &= \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right), \end{aligned} \tag{17}$$

where $0 \le k \le n$. This last product is at most $1/k!$, since each expression in brackets is at most 1, so

$$\left(1 + \frac{1}{n}\right)^n \le 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} = s_n,$$

by equation (16). Thus, by the Limit Inequality Rule for sequences,

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \le \lim_{n\to\infty} s_n = \sum_{n=0}^{\infty}\frac{1}{n!}. \tag{18}$$

On the other hand, if $0 \le m \le n$, then (by equations (16) and (17))

$$\begin{aligned} \left(1 + \frac{1}{n}\right)^n \ge{} & 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots \\ & \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right). \end{aligned}$$

Keeping m fixed and taking limits of the above inequality as $n \to \infty$, we obtain

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \ge 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!} = s_m,$$

by the Limit Inequality Rule. Applying this rule once more to $e \ge s_m$ and using the fact that e is a constant, we obtain

$$e \ge \lim_{m\to\infty} s_m = \sum_{m=0}^{\infty}\frac{1}{m!} = \sum_{n=0}^{\infty}\frac{1}{n!}. \tag{19}$$

Combining inequalities (18) and (19) gives $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!} = e$. ■

By Theorem D37, we can use the series $\sum_{n=0}^{\infty}(x^n/n!)$ as an alternative definition of e^x for all $x \ge 0$. We then define e^x for $x < 0$ as the reciprocal of e^{-x}. For example, $e^{-\pi} = (e^{\pi})^{-1} = 1/e^{\pi}$.

Definition

For $x \geq 0$,

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

For $x < 0$,

$$e^x = (e^{-x})^{-1}.$$

Remarks

1. The equations

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

are also true if x is negative, but we shall not prove this here. The fact that

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}, \quad \text{for } x < 0,$$

is proved in Unit F4 *Power series*.

2. The exponential function $x \longmapsto e^x$ is often called **exp** and we may write

$$\begin{aligned} \exp : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto e^x. \end{aligned}$$

4.2 Calculating e

The representation of e by the infinite series

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}$$

provides a more efficient method of calculating approximate values for e than the equation $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. This is illustrated by the following table of approximate values for $e = 2.718\,281\,828\,45\ldots$.

n	1	2	3	4	5
$(1+1/n)^n$	2	2.25	2.37	2.44	2.49
$\sum_{k=0}^{n}\frac{1}{k!}$	2	2.50	2.67	2.71	2.717

We now estimate how quickly the sequence of partial sums

$$s_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}, \quad n = 1, 2, \ldots,$$

converges to e. The difference between e and s_n is given by

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right) \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right). \end{aligned}$$

The inequality above holds because each term inside the large brackets has been replaced by a larger term. The resulting expression is a geometric series with first term 1 and common ratio $1/(n+1)$, so its sum is

$$\frac{1}{1 - 1/(n+1)} = \frac{n+1}{n}.$$

Hence

$$0 < e - s_n < \frac{1}{(n+1)!} \times \frac{n+1}{n} = \frac{1}{n!} \times \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots. \tag{20}$$

Thus the difference between e and s_n is extremely small when n is large.

Inequality (20) can also be used to show that e is irrational.

Theorem D38

The number e is irrational.

Proof We use proof by contradiction. Suppose that $e = m/n$, where m and n are positive integers. Then, by inequality (20), for this particular integer n we have

$$0 < e - s_n < \frac{1}{n!} \times \frac{1}{n},$$

so

$$0 < n!(e - s_n) < \frac{1}{n}.$$

Since we are assuming that $e = m/n$, we have

$$0 < n!\left(\frac{m}{n} - \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right) < \frac{1}{n}.$$

But the middle expression in this pair of inequalities is an integer, as you can check by multiplying it out, so we have found an integer which lies strictly between 0 and 1, since $1/n \leq 1$ for $n = 1, 2, \ldots$. This is impossible, so e is not rational. ■

Joseph Fourier

Leonhard Euler in 1737 was the first to prove that e is irrational, although his proof, which used continued fractions, was not published until 1744. The first proof by contradiction is due to the French mathematician Joseph Fourier (1768–1830). It appeared in a collection of mathematical results published in 1815 by Fourier's compatriot, Janot de Stainville (1783–1828).

4.3 A fundamental property of e^x

We complete this section by showing that the function $f(x) = e^x$ satisfies one of the Index Laws which we stated in Unit D1. If you are short of time, then you may prefer to skim read the proof of this result.

Theorem D39

For any real numbers x and y, we have $e^{x+y} = e^x e^y$.

Proof First we prove the special case where x and y are both positive. We have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^2}{3!} + \cdots .$$

The following table contains some of the terms which occur when we multiply together partial sums of the power series for e^x and e^y.

	1	y	$\frac{y^2}{2!}$	$\frac{y^3}{3!}$	$\cdots$
1	1	y	$\frac{y^2}{2!}$	$\frac{y^3}{3!}$	$\cdots$
x	x	xy	$\frac{xy^2}{2!}$	$\frac{xy^3}{3!}$	$\cdots$
$\frac{x^2}{2!}$	$\frac{x^2}{2!}$	$\frac{x^2y}{2!}$	$\frac{x^2y^2}{2!2!}$	$\frac{x^2y^3}{2!3!}$	$\cdots$
$\frac{x^3}{3!}$	$\frac{x^3}{3!}$	$\frac{x^3y}{3!}$	$\frac{x^3y^2}{3!2!}$	$\frac{x^3y^3}{3!3!}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Adding the terms on the 'lower left to upper right' diagonals of the table gives:

$$1$$

$$x+y$$

$$\frac{x^2}{2!}+xy+\frac{y^2}{2!}=\frac{(x+y)^2}{2!}$$

$$\frac{x^3}{3!}+\frac{x^2y}{2!}+\frac{xy^2}{2!}+\frac{y^3}{3!}=\frac{(x+y)^3}{3!}$$

$$\vdots$$

$$\frac{x^n}{n!}+\frac{x^{n-1}y}{(n-1)!}+\cdots+\frac{xy^{n-1}}{(n-1)!}+\frac{y^n}{n!}=\frac{(x+y)^n}{n!}$$

For any positive integer n, the product

$$\left(\sum_{k=0}^{n}\frac{x^k}{k!}\right)\left(\sum_{k=0}^{n}\frac{y^k}{k!}\right)=\left(1+x+\cdots+\frac{x^n}{n!}\right)\left(1+y+\cdots+\frac{y^n}{n!}\right)$$

includes *all* the terms in the first $n+1$ diagonals of the table up to the diagonal beginning $\frac{x^n}{n!}$; moreover, all the terms of the product are included in the first $2n+1$ diagonals up to the diagonal containing $\frac{x^n y^n}{n!n!}$, which begins $\frac{x^{2n}}{(2n)!}$.

Since x and y are non-negative, it follows that

$$\sum_{k=0}^{n}\frac{(x+y)^k}{k!}\leq\left(\sum_{k=0}^{n}\frac{x^k}{k!}\right)\left(\sum_{k=0}^{n}\frac{y^k}{k!}\right)\leq\sum_{k=0}^{2n}\frac{(x+y)^k}{k!}.$$

But

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(x+y)^k}{k!}=e^{x+y}\quad\text{and}\quad\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{(x+y)^k}{k!}=e^{x+y}.$$

Thus, by the Squeeze Rule and the Product Rule, we deduce that

$$e^xe^y=\lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{x^k}{k!}\right)\left(\sum_{k=0}^{n}\frac{y^k}{k!}\right)=e^{x+y},$$

as required.

If x and y are not both positive, then we can verify the equation

$$e^xe^y=e^{x+y}$$

by rearranging it so that all the powers are positive (using $e^x=(e^{-x})^{-1}$) and applying the special case just proved. For example, if $x>y>0$, then

$$e^xe^{-y}=e^{x-y}$$

is equivalent to

$$e^x=e^{x-y}e^y,$$

which is true, since $x-y>0$ and $y>0$. ■

Summary

In this unit you have studied infinite series of the form $\sum_{n=1}^{\infty} a_n$. You have seen that such a series is said to be convergent if the sequence of partial sums (s_n) defined by

$$s_n = a_1 + a_2 + \cdots + a_n$$

is convergent, and is said to be divergent otherwise. You have also learnt that it is important to distinguish carefully between the sequence (s_n) and the sequence (a_n). The Non-null Test tells you that, if a series is convergent, then the corresponding sequence (a_n) must be null; but if you know that the sequence (a_n) is null, then this gives you no information about whether or not the series is convergent.

You have met a number of tests that can be used to determine whether a series is convergent and a strategy to help you to work out which test is the most appropriate. If all the terms of a series are non-negative, then you may be able to use the Comparison Test or the Limit Comparison Test to compare with a basic series you know to be convergent or divergent; or you may be able to use the Ratio Test. For a series with infinitely many positive and negative terms, you may be able to use the Absolute Convergence Test or the Alternating Test.

Finally, you have seen how we can use a particular series to define the exponential function e^x, enabling us to prove that e is irrational and that $e^{x+y} = e^x e^y$. In Book F *Analysis 2* you will see how a wide range of functions can be defined using series when you study power series and their properties.

Learning outcomes

After working through this unit, you should be able to:

- explain what is meant by a *convergent* series $\sum_{n=1}^{\infty} a_n$
- write down the sum of a convergent *geometric* series
- find the sums of certain *telescoping* series
- use the Combination Rules for convergent series
- use the Non-null Test to recognise certain *divergent* series
- explain why $\sum_{n=1}^{\infty}(1/n)$ is divergent and $\sum_{n=1}^{\infty}(1/n^2)$ is convergent
- use the Comparison Test, the Limit Comparison Test and the Ratio Test
- recognise and use *basic* series
- explain the term *absolutely convergent* and use the Absolute Convergence Test
- use the Alternating Test
- use the given general strategy for determining whether a series is convergent or divergent
- appreciate that there are two equivalent definitions of e^x
- understand how the series definition of e^x enables us to prove that e is irrational, and that $e^{x+y} = e^x e^y$.

Solutions to exercises

Solution to Exercise D41

(a) Using the formula for summing a finite geometric series, with $a = r = -\frac{1}{3}$, we obtain

$$\begin{aligned} s_n &= \left(-\tfrac{1}{3}\right) + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^3 + \cdots + \left(-\tfrac{1}{3}\right)^n \\ &= \frac{\left(-\frac{1}{3}\right)\left(1 - \left(-\frac{1}{3}\right)^n\right)}{1 - \left(-\frac{1}{3}\right)} \\ &= -\tfrac{1}{4}\left(1 - \left(-\tfrac{1}{3}\right)^n\right). \end{aligned}$$

Since $\left(\left(-\frac{1}{3}\right)^n\right)$ is a basic null sequence,

$$\lim_{n\to\infty} s_n = -\tfrac{1}{4},$$

so

$$\sum_{n=1}^{\infty} \left(-\tfrac{1}{3}\right)^n \text{ is convergent, with sum } -\tfrac{1}{4}.$$

(b) In this case,

$$\begin{aligned} s_n &= (-1) + (-1)^2 + (-1)^3 + \cdots + (-1)^n \\ &= -1 + 1 - 1 + \cdots + (-1)^n. \end{aligned}$$

Thus

$$s_n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Hence the odd subsequence (s_{2k-1}) converges to -1 and the even subsequence (s_{2k}) converges to 0. Thus (s_n) has two convergent subsequences with different limits, so by the First Subsequence Rule (see Unit D2), (s_n) is divergent. It follows that the series $\sum_{n=1}^{\infty}(-1)^n$ is divergent.

(c) In this case, the first term in the series is a_0, so the nth partial sum is the sum of $n+1$ terms. Using the formula for the sum of a finite geometric series with $a = 1$ and $r = \frac{1}{2}$, and summing $n+1$ terms, we obtain

$$\begin{aligned} s_n &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{2}\right)^n \\ &= \frac{1\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)}{1 - \frac{1}{2}} \\ &= 2\left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right) = 2 - \left(\tfrac{1}{2}\right)^n. \end{aligned}$$

Since $\left(\left(\frac{1}{2}\right)^n\right)$ is a basic null sequence,

$$\lim_{n\to\infty} s_n = 2,$$

so

$$\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \text{ is convergent, with sum 2.}$$

You might have anticipated this result, since we proved at the beginning of Subsection 1.1 that

$$\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = 1.$$

Thus, using the fact that $\left(\frac{1}{2}\right)^0 = 1$, we have

$$\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n = 1 + \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = 2.$$

Solution to Exercise D42

We have

$$\begin{aligned} s_1 &= \frac{1}{1\times 2} = \frac{1}{2}, \\ s_2 &= \frac{1}{1\times 2} + \frac{1}{2\times 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}, \\ s_3 &= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}, \\ s_4 &= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}. \end{aligned}$$

Solution to Exercise D43

Using the given identity, we see that

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right).$$

Thus

$$\begin{aligned} s_n &= \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots + \frac{1}{n(n+2)} \\ &= \frac{1}{2}\left(\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)\right. \\ &\quad + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots \\ &\quad \left. + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)\right). \end{aligned}$$

All the terms in alternate brackets cancel, except for the first term in each of the first two brackets and the second term in each of the last two brackets. Thus

$$s_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right).$$

Since $\left(\frac{1}{n+1}\right)$ and $\left(\frac{1}{n+2}\right)$ are null sequences, it follows that

$$\lim_{n\to\infty} s_n = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} \quad \text{is convergent, with sum } \tfrac{3}{4}.$$

Solution to Exercise D44

The series $\sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^n$ is a geometric series with $a = -\frac{3}{4}$ and $r = -\frac{3}{4}$. Since $\left|-\frac{3}{4}\right| = \frac{3}{4} < 1$, the series is convergent, with sum

$$\frac{-\frac{3}{4}}{1 - \left(-\frac{3}{4}\right)} = -\tfrac{3}{7}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, with sum 1; see Subsection 1.2. Hence, by the Combination Rules,

$$\sum_{n=1}^{\infty} \left(\left(-\tfrac{3}{4}\right)^n - \frac{2}{n(n+1)}\right) \quad \text{is convergent,}$$

with sum $-\frac{3}{7} - (2 \times 1) = -\frac{17}{7}$.

Solution to Exercise D45

Let $a_n = \dfrac{(-1)^{n+1} n^2}{2n^2+1}$, so that $|a_n| = \dfrac{n^2}{2n^2+1}$.

By the Combination Rules for sequences,

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \lim_{n\to\infty} \frac{1}{2 + 1/n^2} = \tfrac{1}{2} \neq 0,$$

so the sequence (a_n) is not null.

Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{2n^2+1} \quad \text{is divergent.}$$

Solution to Exercise D46

Let

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

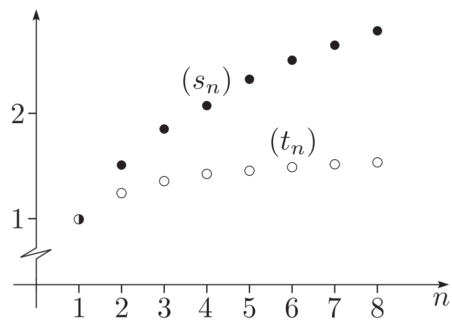
and

$$t_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

The values of the first eight partial sums are given below to two decimal places.

n	1	2	3	4	5	6	7	8
s_n	1	1.5	1.83	2.08	2.28	2.45	2.59	2.72
t_n	1	1.25	1.36	1.42	1.46	1.49	1.51	1.53

The sequences are plotted below.



Solution to Exercise D47

(a) We guess that this series is dominated by $\sum(1/n^3)$. We have

$$n^3 + n \geq n^3 \geq 0, \quad \text{for } n = 1, 2, \ldots,$$

so

$$0 \leq \frac{1}{n^3+n} \leq \frac{1}{n^3}, \quad \text{for } n = 1, 2, \ldots.$$

Since $\sum_{n=1}^{\infty}(1/n^3)$ is convergent, we deduce from the Comparison Test that

$$\sum_{n=1}^{\infty} \frac{1}{n^3+n} \quad \text{is convergent.}$$

(b) Let

$$a_n = \frac{1}{n + \sqrt{n}}, \quad \text{for } n = 1, 2, \ldots.$$

We guess that the terms of this series behave like $1/n$ for large n, and we know that $\sum_{n=1}^{\infty}(1/n)$ is divergent. We cannot compare a_n and $1/n$ directly, so we use the Limit Comparison Test with

$$b_n = \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots.$$

Both a_n and b_n are positive and

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{1}{n+\sqrt{n}} \times \frac{n}{1}\\ &= \frac{n}{n+\sqrt{n}}\\ &= \frac{1}{1+1/\sqrt{n}} \to 1 \neq 0.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1/n)$ is divergent, we deduce by the Limit Comparison Test that

$$\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} \text{ is divergent.}$$

(c) Let

$$a_n = \frac{n+4}{2n^3-n+1}, \quad \text{for } n = 1, 2, \ldots.$$

We guess that the terms of this series behave like $n/n^3 = 1/n^2$ for large n, and we know that $\sum_{n=1}^{\infty}(1/n^2)$ is convergent. We cannot compare a_n and $1/n^2$ directly, so we use the Limit Comparison Test with

$$b_n = \frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Both a_n and b_n are positive and

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{n+4}{2n^3-n+1} \times \frac{n^2}{1}\\ &= \frac{n^3+4n^2}{2n^3-n+1}\\ &= \frac{1+4/n}{2-1/n^2+1/n^3} \to \frac{1}{2} \neq 0.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1/n^2)$ is convergent, we deduce by the Limit Comparison Test that

$$\sum_{n=1}^{\infty} \frac{n+4}{2n^3-n+1} \text{ is convergent.}$$

(d) We guess that this series is dominated by $\sum_{n=1}^{\infty}(1/n^3)$. Indeed, since

$$0 \leq \cos^2(2n) \leq 1, \quad \text{for } n = 1, 2, \ldots,$$

we have

$$0 \leq \frac{\cos^2(2n)}{n^3} \leq \frac{1}{n^3}, \quad \text{for } n = 1, 2, \ldots.$$

Since $\sum_{n=1}^{\infty}(1/n^3)$ is convergent, we deduce by the Comparison Test that

$$\sum_{n=1}^{\infty} \frac{\cos^2(2n)}{n^3} \text{ is convergent.}$$

Solution to Exercise D48

(a) Let

$$a_n = \frac{n^3}{n!}, \quad n = 1, 2, \ldots.$$

Then a_n is positive and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^3}{(n+1)!} \times \frac{n!}{n^3}\\ &= \frac{(n+1)^3}{(n+1)n^3}\\ &= \frac{n^2+2n+1}{n^3} = \frac{1}{n} + \frac{2}{n^2} + \frac{1}{n^3}.\end{aligned}$$

Hence, by the Combination Rules for sequences,

$$\frac{a_{n+1}}{a_n} \to 0 \text{ as } n \to \infty.$$

Thus, by the Ratio Test,

$$\sum_{n=1}^{\infty} \frac{n^3}{n!} \text{ is convergent.}$$

(b) Let

$$a_n = \frac{n^2 2^n}{n!}, \quad n = 1, 2, \ldots.$$

Then a_n is positive and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^2 2^{n+1}}{(n+1)!} \times \frac{n!}{n^2 2^n}\\ &= \frac{2(n+1)^2}{(n+1)n^2} = 2\left(\frac{1}{n} + \frac{1}{n^2}\right).\end{aligned}$$

Hence, by the Combination Rules for sequences,

$$\frac{a_{n+1}}{a_n} \to 0 \text{ as } n \to \infty.$$

Thus, by the Ratio Test,

$$\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!} \text{ is convergent.}$$

(c) Let

$$a_n = \frac{(2n)!}{n^n}, \quad n = 1, 2, \ldots.$$

Then a_n is positive and

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2(n+1))!}{(n+1)^{n+1}} \times \frac{n^n}{(2n)!}\\ &= \frac{(2n+2)!\, n^n}{(n+1)^{n+1}(2n)!}\\ &= \frac{(2n+2)(2n+1)n^n}{(n+1)^{n+1}}\\ &= \frac{2(2n+1)n^n}{(n+1)^n} = \frac{4n+2}{(1+1/n)^n}.\end{aligned}$$

Now

$$\lim_{n\to\infty}(1+1/n)^n = e \quad \text{and} \quad \lim_{n\to\infty}\frac{1}{4n+2} = 0,$$

so $((1+1/n)^n/(4n+2))$ is null, by the Product Rule.

We deduce by the Reciprocal Rule that

$$\frac{a_{n+1}}{a_n} \to \infty \quad \text{as } n \to \infty;$$

see Subsection 4.3 of Unit D2.

Hence, by the Ratio Test,

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^n} \text{ is divergent.}$$

(Alternatively, note that

$$\begin{aligned}\frac{(2n)!}{n^n} &\geq \left(\frac{2n}{n}\right)\left(\frac{2n-1}{n}\right)\cdots\left(\frac{n+1}{n}\right)\\ &\geq 1,\end{aligned}$$

so, by the Non-null Test,

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n^n} \text{ is divergent.)}$$

Solution to Exercise D49

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ is a basic convergent series of type (d), with $c = 2$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ is a basic convergent series of type (a), with $p = \frac{5}{2}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ is a basic divergent series of type (e), with $p = \frac{2}{3}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ is a basic convergent series of type (c), with $p = 1$ and $c = \frac{1}{2}$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n}$ is a basic convergent series of type (b), with $c = \frac{1}{4}$.

Solution to Exercise D50

(a) Let

$$a_n = \frac{(-1)^{n+1}n}{n^3+1}, \quad n = 1, 2, \ldots.$$

Then

$$|a_n| = \frac{n}{n^3+1}, \quad \text{for } n = 1, 2, \ldots.$$

Now

$$\frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots,$$

and $\sum_{n=1}^{\infty}(1/n^2)$ is a basic convergent series. Hence, by the Comparison Test,

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} \text{ is convergent.}$$

By the Absolute Convergence Test, it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^3+1} \text{ is convergent.}$$

(b) If we write this series as $\displaystyle\sum_{n=0}^{\infty} a_n$, then

$$|a_n| = \frac{1}{2^n}, \text{ for } n = 0, 1, 2, \ldots.$$

Since $\sum_{n=0}^{\infty}(1/2^n)$ is a basic convergent series, it follows from the Absolute Convergence Test that

$$\sum_{n=0}^{\infty} a_n \text{ is convergent.}$$

Solution to Exercise D51

If we write the series as $\sum_{n=1}^{\infty} a_n$, then

$$|a_n| = \frac{1}{2^n}, \quad \text{for } n = 1, 2, \ldots.$$

Since $\sum_{n=1}^{\infty}(1/2^n)$ is a basic convergent (geometric) series, it follows from the Absolute Convergence Test that $\sum_{n=1}^{\infty} a_n$ is convergent.

By the infinite form of the Triangle Inequality, we have

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1.$$

Hence the sum lies in the interval $[-1, 1]$.

Solution to Exercise D52

(a) Let

$$a_n = \frac{(-1)^{n+1}}{n^{1/3}}, \quad \text{for } n = 1, 2, \ldots.$$

Then $a_n = (-1)^{n+1} b_n$, where

$$b_n = \frac{1}{n^{1/3}}, \quad \text{for } n = 1, 2, \ldots.$$

Now

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. (b_n) is a basic null sequence
3. (b_n) is decreasing, because $(1/b_n) = (n^{1/3})$ is increasing.

Hence, by the Alternating Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/3}} \text{ is convergent.}$$

(b) Let

$$a_n = \frac{(-1)^{n+1}}{n + n^{1/2}}, \quad \text{for } n = 1, 2, \ldots.$$

Then $a_n = (-1)^{n+1} b_n$, where

$$b_n = \frac{1}{n + n^{1/2}}, \quad \text{for } n = 1, 2, \ldots.$$

Now

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. (b_n) is a null sequence by the Squeeze Rule, since
$$0 \leq \frac{1}{n + n^{1/2}} \leq \frac{1}{n}, \quad \text{for } n = 1, 2, \ldots$$
3. (b_n) is decreasing, because $(1/b_n) = (n + n^{1/2})$ is increasing.

Hence, by the Alternating Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + n^{1/2}} \text{ is convergent.}$$

(c) Let

$$a_n = \frac{(-1)^{n+1} n}{n+2}, \quad \text{for } n = 1, 2, \ldots.$$

Then

$$|a_n| = \frac{n}{n+2} = \frac{1}{1 + 2/n} \to 1 \neq 0.$$

Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+2} \text{ is divergent.}$$

(Notice that if you try to apply the Alternating Test here, by writing $a_n = (-1)^{n+1} b_n$, where

$$b_n = \frac{n}{n+2}, \quad \text{for } n = 1, 2, \ldots,$$

then you find that (b_n) is not null, so the Alternating Test cannot be used. Indeed, the series $\sum_{n=1}^{\infty} a_n$ is divergent, as the Non-null Test shows.)

Solution to Exercise D53

(a) We have

$$\frac{5n + 2^n}{3^n} = 5n\left(\tfrac{1}{3}\right)^n + \left(\tfrac{2}{3}\right)^n, \quad \text{for } n = 1, 2, \ldots.$$

Now

$$\sum_{n=1}^{\infty} n\left(\tfrac{1}{3}\right)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\tfrac{2}{3}\right)^n$$

are both basic convergent series. Thus, by the Combination Rules for series,

$$\sum_{n=1}^{\infty} \frac{5n + 2^n}{3^n} \text{ is convergent.}$$

(b) We guess that the terms of the series behave like $1/n^3$ for large n, so we use the Limit Comparison Test with

$$a_n = \frac{3}{2n^3 - 1}, \quad \text{for } n = 1, 2, \ldots$$

and

$$b_n = 1/n^3, \quad \text{for } n = 1, 2, \ldots.$$

Both a_n and b_n are positive and

$$\begin{aligned}\frac{a_n}{b_n} &= \frac{3}{2n^3-1} \times \frac{n^3}{1}\\ &= \frac{3n^3}{2n^3-1} = \frac{3}{2-1/n^3} \to \frac{3}{2} \neq 0.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(1/n^3)$ is a basic convergent series, we deduce from the Limit Comparison Test that

$$\sum_{n=1}^{\infty} \frac{3}{2n^3-1} \text{ is convergent.}$$

(c) We use the Alternating Test.

Let

$$a_n = \frac{(-1)^{n+1}}{n\log(n+1)}, \quad \text{for } n = 1, 2, \ldots.$$

Then $a_n = (-1)^{n+1} b_n$, where

$$b_n = \frac{1}{n\log(n+1)}, \quad \text{for } n = 1, 2, \ldots.$$

Now

1. $b_n \geq 0$, for $n = 1, 2, \ldots$
2. (b_n) is a null sequence, by the Squeeze Rule, since
$$0 \leq \frac{1}{n\log(n+1)} \leq \frac{1}{n\log 2}, \quad \text{for } n = 1, 2, \ldots$$
3. (b_n) is decreasing, because $(1/b_n) = (n\log(n+1))$ is increasing.

Hence, by the Alternating Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\log(n+1)} \text{ is convergent.}$$

(d) Let

$$a_n = \frac{(-1)^{n+1}n^2}{n^2+1}, \quad \text{for } n = 1, 2, \ldots.$$

Then

$$|a_n| = \frac{n^2}{n^2+1} = \frac{1}{1+1/n^2} \to 1 \neq 0.$$

Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

(e) Let

$$a_n = \frac{(-1)^{n+1}n}{n^3+5}, \quad \text{for } n = 1, 2, \ldots.$$

Then

$$|a_n| = \frac{n}{n^3+5}, \quad \text{for } n = 1, 2, \ldots.$$

Thus

$$|a_n| \leq \frac{n}{n^3} = \frac{1}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Since $\sum(1/n^2)$ is a basic convergent series, we deduce by the Comparison Test that

$$\sum_{n=1}^{\infty} |a_n| \text{ is convergent.}$$

Hence, by the Absolute Convergence Test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{n^3+5} \text{ is convergent.}$$

(f) Let

$$a_n = \frac{2^n}{n^6}, \quad \text{for } n = 1, 2, \ldots.$$

Then a_n is positive and

$$\frac{1}{a_n} = \frac{n^6}{2^n} \to 0,$$

since $(n^6/2^n)$ is a basic null sequence. So, by the Reciprocal Rule, $a_n \to \infty$. Hence, by the Non-null Test,

$$\sum_{n=1}^{\infty} \frac{2^n}{n^6} \text{ is divergent.}$$